Multiple imputation: Isn‘t Rubin’s estimator over-optimistic?

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Plan

1. Multiple imputation and Rubin's estimator
2. GmG model
3. The Rubin-type family of estimators
4. Unbiased Rubin-type estimators

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Imputation estimators

$\mathbf{X} = (X_1, \ldots, X_n)$ - the sample:

- $\mathbf{X}_R = (X_i,\ i \in R)$ - observed part of $\mathbf{X}$,
- $\mathbf{X}_{R^c} = (X_i,\ i \in R^c)$ - missing part of $\mathbf{X}$.

Missing values are replaced by imputed: $\widetilde{X}_i$, $i \in R^c$, i.e. the imputed sample $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ is defined through

$$\widetilde{X}_i = \begin{cases} X_i, & \text{if } i \in R, \\ \widetilde{X}_i \text{ (imputed)}, & \text{if } i \in R^c. \end{cases}$$

Let $\hat{\theta} := h(\mathbf{x})$ be an estimator of parameter θ. Its imputation version is

$$\hat{\theta}_{\mathrm{Imp}} = h(\widetilde{\mathbf{X}}).$$

The imputation version of $\bar{X}$ and S^2:

$$\bar{X}_{Imp} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i, \qquad S^2_{Imp} = \frac{1}{n-1}\sum_{i=1}^{n} (\widetilde{X}_i - \bar{X}_{Imp})^2.$$

Multiple imputation (MI) estimators

Multiple imputation:

- imputed samples

$$\widetilde{\mathbf{X}}^{(\ell)} = (X_i,\ i \in R,\ \widetilde{X}_i^{(\ell)},\ i \in R^c), \quad \ell = 1, \dots, m,$$

- respective estimators

$$\hat{\theta}^{(\ell)}_{\mathrm{Imp}} = h\left(\widetilde{\mathbf{X}}^{(\ell)}\right), \quad \bar{X}^{(\ell)}_{\mathrm{Imp}}, \quad (S^{(\ell)}_{\mathrm{Imp}})^2, \quad \ell = 1, \dots, m.$$

MImp estimators

$$\hat{\theta}_{\mathrm{MImp}} = \frac{1}{m}\sum_{\ell=1}^{m} \hat{\theta}^{(\ell)}_{\mathrm{Imp}}, \quad \bar{X}_{\mathrm{MImp}} = \frac{1}{m}\sum_{\ell=1}^{m} \bar{X}^{(\ell)}_{\mathrm{Imp}}.$$

Rubin's estimator of the variance of $\bar{X}_{\text{MImp}}$

The popular Rubin estimator of the variance of $\bar{X}_{\text{MImp}}$ is

$$\hat{\nu}^2_{\text{Rubin}} = \bar{U}_m + \tfrac{m+1}{m} B_m,$$

where

$$\bar{U}_m = \tfrac{1}{mn} \sum_{\ell=1}^{m} (S^{(\ell)}_{\text{Imp}})^2$$

and

$$B_m = \tfrac{1}{m-1} \sum_{\ell=1}^{m} (\bar{X}^{(\ell)}_{\text{Imp}} - \bar{X}_{\text{MImp}})^2.$$

Over-optimistic...?

Let $(\mathbf{X}_R, \mathbf{X}_{R^c})$ be an iid sample with mean μ and variance σ^2.

Hot-deck imputation: $\widetilde{X}_j^{(\ell)} = X_{K_j^{(\ell)}}$, $j \in R^c$, where $K_j^{(\ell)}$ are iid uniform on R, $\ell = 1, \ldots, m$, $j \in R^c$,

Then

$$\bar{X}_{\mathrm{MImp}} = f\bar{X}_R + \frac{1}{mn} \sum_{\ell=1}^{m} \sum_{j \in R^c} X_{K_j^{(\ell)}}$$

is unbiased for μ and $\mathbb{V}\mathrm{ar}(\bar{X}_{\mathrm{MImp}}) = \frac{\sigma^2}{r}\left(1 + \frac{(1-f)(r-1)}{mn}\right)$

But

$$\mathbb{B}(\nu^2_{\mathrm{Rubin}}) = \mathbb{E}\,\nu^2_{\mathrm{Rubin}} - \mathbb{V}\mathrm{ar}(\bar{X}_{\mathrm{MImp}}) = -\frac{(1-f)[n(n-r+1)+2(r-1)]}{n(n-1)r} < \mathbf{0}.$$

Over-optimistic...?

Let (X_R, X_{R^c}) be a sample from $\mathrm{N}(\mu, \sigma^2)$.

Hot-deck imputation: $\widetilde{X}_j^{(\ell)} = \bar{X}_R + S_R Z_j^{(\ell)}$, where $\bar{X}_R$ and S_R^2 are the sample ($\mathbf{X}_R$) mean and variance and $Z_j^{(\ell)}$, $\ell = 1, \ldots, m$, $j \in R^c$, are iid $\mathrm{N}(0,1)$ and are independent of $\mathbf{X}_R$.

Then

$$\bar{X}_{\mathrm{MImp}} = \bar{X}_R + \tfrac{1-f}{mn} S_R \sum_{\ell=1}^{m} \sum_{j \in R^c} Z_j^{(\ell)}$$

is unbiased and $\mathbb{V}\mathrm{ar}(\bar{X}_{\mathrm{MImp}}) = \frac{\sigma^2}{r}\left(1 + \frac{(1-f)f}{m}\right)$.

But

$$\mathbb{B}(\nu^2_{\mathrm{Rubin}}) = \mathbb{E}\,\nu^2_{\mathrm{Rubin}} - \mathbb{V}\mathrm{ar}(\bar{X}_{\mathrm{MImp}}) = -\frac{\sigma^2(1-f)[(n-1)(n-r)-1]}{n(n-1)r} < \mathbf{0}.$$

Deficiencies of Rubin's estimator

In both cases:

$$\frac{\mathbb{B}(\nu^2_{\mathrm{Rubin}})}{\mathbb{V}\mathrm{ar}(\bar{X}_{\mathrm{MImp}})} \xrightarrow{n\to\infty} -1.$$

Basic problems with $\hat{\nu}^2_{\mathrm{Rubin}}$:

Typically it is

- **BIASED**
- **NON-ADMISSIBLE**

We will discuss these issues in a relatively simple Bayesian GmG-model: Gaussian variables with Gaussian mean.

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GmG Bayesian model:

Let $\mu \in \mathbb{R}$, $\kappa, \sigma > 0$. Then the $\mathrm{GmG}(\mu, \sigma^2, \kappa)$ Bayesian model is defined through:

$$\mathbf{X}|M \sim \left(\mathrm{N}(M, \sigma^2)\right)^{\otimes n},$$

$$M \sim \mathrm{N}(\mu, \kappa\sigma^2).$$

Then

$$\mathbf{X}_{R^c}|\mathbf{X}_R \sim \mathrm{N}\left(\frac{r\kappa\bar{X}_R+\mu}{r\kappa+1},\ \sigma^2(\mathbb{I}_{R_c} + \frac{\kappa}{r\kappa+1}\mathbf{1}_{R_c}\mathbf{1}_{R_c}^T)\right). \qquad (1)$$

Non-informative prior: for $\kappa = \infty$. Take $\kappa \to \infty$ in (1):

$$\mathbf{X}_{R^c}|\mathbf{X}_R \sim \mathrm{N}\left(\bar{X}_R,\ \sigma^2(\mathbb{I}_{R_c} + \frac{1}{r}\mathbf{1}_{R_c}\mathbf{1}_{R_c}^T)\right). \qquad (2)$$

Representation

Consequently, $\mathbf{X}_{R^c}$ has the representation

$$\mathbf{X}_{R^c} = \frac{\kappa r \bar{X}_R + \mu}{\kappa r + 1} \mathbf{1}_{R^c} + \sigma \left(\mathbf{Z} + \sqrt{\frac{\kappa}{\kappa r + 1}}\, V\, \mathbf{1}_{R^c} \right), \qquad (3)$$

where $r = \#(R)$,

- $\mathbf{Z} = (Z_i,\ i \in R^c)$ has iid $\mathrm{N}(0,1)$ components,
- $V \sim \mathrm{N}(0,1)$,
- $(\mathbf{Z},\ V,\ \mathbf{X}_R)$ independent.

Assume σ^2 unknown. How to impute?

Approximate

$$\sigma^2 \approx S_R^2 = \frac{1}{r-1} \sum_{k \in R} (X_k - \bar{X}_R)^2.$$

Impute missing X's by

$$\widetilde{\mathbf{X}}_{R^c}^{(\ell)} = \frac{\kappa r \bar{X}_R + \mu}{\kappa r + 1} \mathbf{1}_{R^c} + S_R \left(\mathbf{Z}^{(\ell)} + \sqrt{\frac{\kappa}{\kappa r + 1}} V^{(\ell)} \mathbf{1}_{R^c} \right),$$

$\ell = 1, \ldots, m$.

Here, $(\mathbf{Z}^{(\ell)}, V^{(\ell)})$, $\ell = 1, \ldots$, are independent copies of $(\mathbf{Z}, V)$.
In particular, $(\mathbf{Z}^{(\ell)}, V^{(\ell)})_{\ell=1,\ldots,m}$ and $\mathbf{X}_R$ are independent.

The case of non-informative prior

The case of non-informative prior is by taking $\kappa \to \infty$ in (3).

Then

$$\mathbf{X}_{R^c} = \bar{X}_R \mathbf{1}_{R^c} + \sigma \left(\mathbf{Z} + \frac{1}{\sqrt{r}} V \mathbf{1}_{R^c} \right),$$

and thus impute missing X's by

$$\widetilde{\mathbf{X}}^{(\ell)}_{R^c} = \bar{X}_R \mathbf{1}_{R^c} + S_R \left(\mathbf{Z}^{(\ell)} + \frac{1}{\sqrt{r}} V^{(\ell)} \mathbf{1}_{R^c} \right),$$

$\ell = 1, \ldots, m$.

Theorem

Consider the $\mathrm{GmG}(\mu, \sigma^2, \kappa)$ *model. The multiple imputation estimator of M has the form*

$$\bar{X}_{MImp} = f\,\frac{\kappa\, n+1}{\kappa\, r+1}\bar{X}_R + (1-f)\left(\frac{\mu}{\kappa\, r+1} + S_R \underline{W}\right), \qquad (4)$$

where $f = \frac{r}{n}$ *and* $\underline{W} = \frac{1}{m}\sum_{\ell=1}^{m} \bar{W}^{(\ell)}$ *with*

$$\bar{W}^{(\ell)} = \frac{1}{n-r}\sum_{i\in R^c} Z_i^{(\ell)} + \sqrt{\frac{\kappa}{\kappa\, r+1}}\, V^{(\ell)}.$$

$\bar{X}_{\mathrm{MImp}}$ *is ubiased for M, i.e.* $\mathbb{E}\,\bar{X}_{\mathrm{MImp}} = \mathbb{E}\,M$*, and*

$$\mathrm{MSE}\,\bar{X}_{\mathrm{MImp}} = \mathbb{E}\left(\bar{X}_{\mathrm{MImp}} - M\right)^2 = \left(\frac{n\kappa+f}{n\kappa+1} + \frac{1-f}{m}\right)\frac{\tau^2\sigma^2}{r}, \qquad (5)$$

where $\tau^2 = \frac{n\kappa+1}{r\kappa+1}\, f$*.*

Theorem (cont.)

Statistics B_m and $\bar{U}_m$ assume the form:

$$
\begin{aligned}
B_m =& (1-f)^2\, S_R^2\, S_{\bar{W}}^2, \\
\bar{U}_m =& \frac{1}{n(n-1)} \left\{ S_R^2 \left[r - 1 + (n-r-1)\bar{S}_Z^2 \right] \right. \\
& \left. + r(1-f)\left(\frac{\bar{X}_R - \mu}{r\kappa + 1} - S_R \underline{W} \right)^2 + \frac{r}{1-f}\, \frac{m-1}{m}\, B_m \right\},
\end{aligned}
$$

where

$$
\bar{S}_Z^2 = \frac{1}{m} \sum_{\ell=1}^{m} S_{Z^{(\ell)}}^2 \quad \textit{and} \quad S_{\bar{W}}^2 = \frac{1}{m-1} \sum_{\ell=1}^{m} \left(\bar{W}^{(\ell)} - \underline{W} \right)^2 .
$$

Moreover,

$$
\mathbb{E}\, \bar{U}_m = \frac{\sigma^2}{n}, \quad \textit{and} \quad \mathbb{E}\, B_m = (1-f)\frac{\tau^2 \sigma^2}{r}.
$$

Theorem (cont.)

Rubin's estimator ν^2_{Rubin} of $\text{MSE}(\bar{X}_{\text{MImp}})$ is biased with the bias

$$\mathbb{B}\,\nu^2_{\text{Rubin}} = \mathbb{E}\,\nu^2_{\text{Rub}} - \text{MSE}(\bar{X}_{\text{MImp}}) = \frac{2(1-f)}{\kappa\, n+1}\,\frac{\tau^2\sigma^2}{r}.$$

The relative bias of Rubin's estimator has the form

$$\frac{\mathbb{B}\,\nu^2_{\text{Rubin}}}{\text{MSE}(\bar{X}_{\text{MImp}})} = \frac{2(1-f)}{\kappa\, n+f+\frac{1}{m}(1-f)(\kappa\, n+1)} < \frac{2(1-f)}{\kappa\, n+f}. \qquad (6)$$

Theorem (cont.)

For non-informative prior, i.e. when $\kappa \to \infty$:

- $\mathrm{MSE}(\bar{X}_{MImp}) = \mathbb{E}\left(\bar{X}_{MImp} - M\right)^2 = \frac{\sigma^2}{r}\left(1 + \frac{1-f}{m}\right)$;
- *Rubin's estimator, ν^2_{Rubin}, is unbiased for $\mathrm{MSE}(\bar{X}_{\mathrm{MImp}})$.*

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The Rubin-type family

Recall

$$\bar{U}_m = \frac{1}{mn}\sum_{\ell=1}^{m}(S_{\mathrm{Imp}}^{(\ell)})^2 \quad \text{and} \quad B_m = \frac{1}{m-1}\sum_{\ell=1}^{m}(\bar{X}_{\mathrm{Imp}}^{(\ell)} - \bar{X}_{\mathrm{MImp}})^2.$$

We introduce the Rubin-type family of estimators of the $\mathrm{MSE}(\bar{X}_{\mathrm{MImp}})$:

$$\mathfrak{R} = \left\{\nu^2(\alpha,\beta) = \alpha\bar{U}_m + \beta B_m, \quad \alpha,\beta \in \mathbb{R}\right\}.$$

Examples

- Rubin's estimator:

$$\nu^2_{\text{Rubin}} = \nu^2_{1,\frac{m+1}{m}} \in \mathfrak{R},$$

  i.e. it is Rubin-type with $\alpha = 1$ and $\beta = \frac{m+1}{m}$.

- Bjørnstad's estimator:

$$\nu^2_{\text{Bjørnstad}} = \nu^2_{1,\frac{m+1-f}{m(1-f)}} \in \mathfrak{R},$$

  i.e. it is Rubin-type with $\alpha = 1$ and $\beta = \frac{m+1-f}{m(1-f)} \stackrel{f\to 0}{\to} \frac{m+1}{m}$

Towards optimal weights

We search for the optimal estimator of the MSE of $\bar{X}_{\mathrm{MImp}}$ within the family $\mathcal{R}$, i.e. we serach for optimal weights (α, β).

The result is elementary but the formulas are complicated.

Monster constants

We need to introduce three + four constants

$$a = 1 + \frac{2n(1-f)}{(n-1)^2 m} + \frac{2(1-\tau^2)\left(4-2n+\left(r-2-\frac{r+1}{m}\right)(1+\tau^2)\right)}{(r+1)(n-1)^2},$$

$$b = \frac{\tau^4(1-f)^2}{f^2}\,\frac{m+1}{m-1},$$

$$c = \frac{\tau^2(1-f)}{f}\left(1 + \frac{2\tau^2}{m(n-1)} + \frac{2(\tau^2-1)}{(r+1)(n-1)}\right)$$

$$A_1 = a\,\frac{m+1}{m-1} - \left(1 + \frac{2\tau^2}{m(n-1)} + \frac{2(\tau^2-1)}{(r+1)(n-1)}\right)^2,$$

$$A_2 = \frac{1}{m-1} - \frac{\tau^2}{m(n-1)} + \frac{1-\tau^2}{(r+1)(n-1)},$$

$$A_3 = \frac{\tau^2-r}{m} + (1-\tau^2)\left(\frac{n}{m} + \frac{3-n+\left(r-2-\frac{r+1}{m}\right)(1+\tau^2)}{r+1}\right),$$

$$A_4 = \tau^2 + \frac{(1-f)\tau^2}{m} - (1-\tau^2)f.$$

Theorem

Let

$$\alpha_* = \frac{2n(r-1)}{r(r+1)} \frac{A_2 A_4}{A_1} \qquad \textit{and} \qquad \beta_* = \frac{2(r-1)}{(n-1)^2(r+1)(1-f)\tau^2} \frac{A_3 A_4}{A_1}.$$

Then $\nu^2(\alpha_*, \beta_*)$ *has the smallest MSE among the estimators of the* $\mathrm{MSE}(\bar{X}_{MImp})$ *in the Rubin-type family* $\mathfrak{R}$*.*

The optimal MSE is

$$\begin{aligned}&\mathrm{MSE}(\nu^2(\alpha_*, \beta_*)) \\ =&\frac{\sigma^4}{n^2}\left[\frac{(r+1)(\alpha_*^2 a+\beta_*^2 b+2\alpha_*\beta_* c)}{r-1} - 2\alpha_* \frac{A_4}{f} - 2\beta_* \frac{(1-f)\tau^2 A_4}{f^2} + \frac{A_4^2}{f^2}\right].\end{aligned}$$

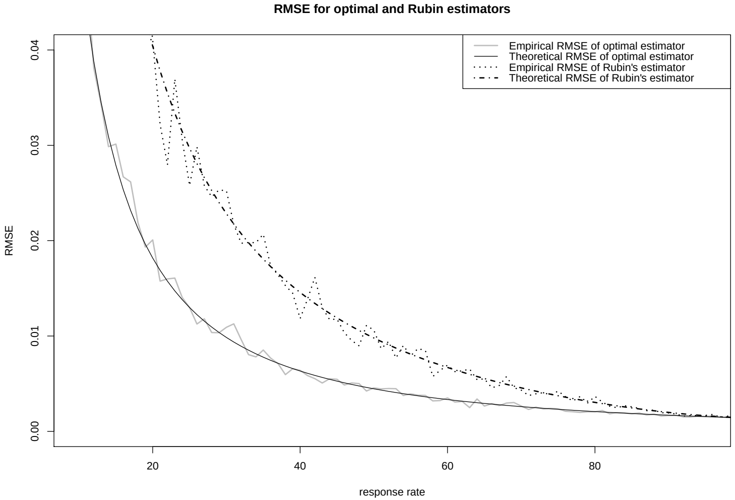

Figure: The RMSE of the optimal $\nu^2(\alpha_*, \beta_*)$ estimator is smaller that the RMSE of Rubin's estimator. Here $m = 5$, $n = 100$, $\sigma^2 = 1$, $\mu = 0$ and $\kappa = 1$. The empirical versions are computed from 100 repetitions.

Theorem (Non-informative prior)

Consider the case of $\kappa \to \infty$, i.e. $\tau = 1$. Let

$$\alpha_{*,\infty} = \frac{nm-2m+1}{f(m-1)}K \quad \text{and} \quad \beta_{*,\infty} = -\frac{r-1}{(1-f)(n-1)}K, \tag{7}$$

where

$$K = \frac{2(r-1)\left(1+\frac{1-f}{m}\right)}{m(n-1)(r+1)\left[\left(1+\frac{2(n-r)}{m(n-1)^2}\right)\left(1+\frac{2}{m-1}\right)-\left(1+\frac{2}{m(n-1)}\right)^2\right]}.$$

Then $\nu^2(\alpha_{*,\infty}, \beta_{*,\infty})$ is the optimal estimator of $\mathrm{MSE}(\bar{X}_{MImp})$ in the family $\mathfrak{R}$. Its MSE is

$$\begin{aligned}
&\mathrm{MSE}\left(\nu^2(\alpha_{*,\infty}, \beta_{*,\infty})\right)\\
&= \frac{\sigma^4}{n^2}\left\{\frac{r+1}{r-1}\left[\alpha_{*,\infty}^2\left(1+\frac{2n(1-f)}{(n-1)^2 m}\right) + \beta_{*,\infty}^2\frac{(1-f)^2}{f^2}\left(1+\frac{2}{m-1}\right)\right.\right.\\
&\left.+2\alpha_{*,\infty}\beta_{*,\infty}\frac{(1-f)}{f}\left(1+\frac{2}{m(n-1)}\right)\right]\\
&\left.-\left[2\alpha_{*,\infty}f + 2\beta_{*,\infty}(1-f)^2 - \left(1+\frac{1-f}{m}\right)\right]\frac{1+\frac{1-f}{m}}{f^2}\right\}.
\end{aligned}$$

Simplified quasi-optimal weights

Since

$$\lim_{n\to\infty} \alpha_{*,\infty} = \frac{1+\frac{1-f}{m}}{f} \quad \text{and} \quad \lim_{n\to\infty} n\beta_{*,\infty} = -\frac{(m-1)f\left(1+\frac{1-f}{m}\right)}{m(1-f)}$$

if the sample size n is large and number of imputations m is small one can use approximate values of $\alpha_{*,\infty}$ and $\beta_{*,\infty}$:

$$\widetilde{\alpha}_{*,\infty} = \frac{1+\frac{1-f}{m}}{f} \quad \text{and} \quad \widetilde{\beta}_{*,\infty} = -\frac{(m-1)f\left(1+\frac{1-f}{m}\right)}{nm(1-f)}.$$

Since

$$\lim_{m\to\infty} \widetilde{\alpha}_{*,\infty} = \frac{1}{f} \quad \text{and} \quad \lim_{m\to\infty} n\widetilde{\beta}_{*,\infty} = -\frac{f}{1-f}$$

if both n and m are large one can use approximate values of $\alpha_{*,\infty}$ and $\beta_{*,\infty}$:

$$\alpha_{**} = \frac{1}{f} \qquad \text{and} \qquad \beta_{**} = -\frac{f}{n(1-f)}.$$

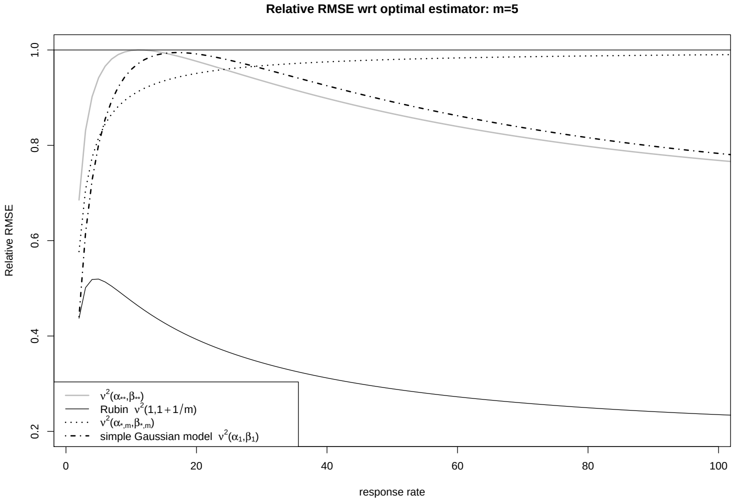

Figure: Simplified and approximate optimal estimators of the MSE perform better than Rubin's estimator. The computations were done for $m = 5$, $n = 500$, $\sigma^2 = 1$, $\kappa = \infty$ $(\tau^2 = 1)$.

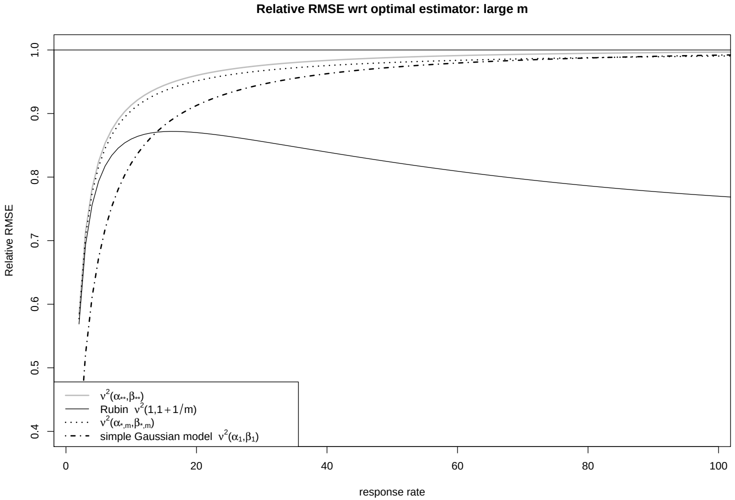

Figure: Simplified and approximate optimal estimators of the MSE still perform better than Rubin's estimator. The computations were done for $m = 100$, $n = 500$, $\sigma^2 = 1$, $\kappa = \infty$ ($\tau^2 = 1$).

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Unbiased Rubin-type estimators

Rubin's estimator is unbiased for $\mathrm{MSE}(\bar{X}_{\mathrm{MImp}})$, i.e.

$$\nu^2_{\mathrm{Rubin}} \in \mathfrak{R}_u = \{\nu^2_{\alpha,\beta} \in \mathfrak{R} : \ \mathbb{E}\,\nu^2_{\alpha,\beta} = \mathrm{MSE}\,\bar{X}_{MImp}\}.$$

As it is shown below, ν^2_{Rubin} is non-admissible also in $\mathfrak{R}_u$.

Theorem

Let

$$\alpha_{*,u} = \tfrac{1}{f}\left(1 + \tfrac{1-f}{m}\right) \frac{(m(n-2)+1)(n-1)}{m(n-1)^2-(m-1)(n+r-2)} \tag{8}$$

and

$$\beta_{*,u} = -\tfrac{1}{1-f}\left(1 + \tfrac{1-f}{m}\right) \frac{(r-1)(m-1)}{m(n-1)^2-(m-1)(n+r-2)}. \tag{9}$$

Then $\nu^2(\alpha_{*,u}, \beta_{*,u})$ is optimal estimator of the MSE of the $\bar{X}_{MImp}$ in the class $\mathfrak{R}_u$.

Simplified versions of $\nu^2(\alpha_{*,u},\ \beta_{*,u})$ for large n and small/large m are the same as for $\nu^2(\alpha_{*,\infty},\ \beta_{*,\infty})$.

MSE of Rubin's estimator

Theorem

$$\mathrm{MSE}(\nu^2_{Rub}) = \frac{2\sigma^4}{r-1}\left(\left[\frac{1}{n} + \frac{(m+1)(1-f)}{mr}\right]^2 + A\right), \qquad (10)$$

where

$$A = \frac{r+1}{m}\left[\frac{n-r}{n^2(n-1)^2} + \frac{(m+1)^2(1-f)^2}{m(m-1)r^2} + 2\frac{(m+1)(1-f)}{mrn(n-1)}\right].$$

Non-admissibility of Rubin's estimator

Now we will compare MSE of the optimal unbiased estimator $\nu^2(\alpha_{*,u}, \beta_{*,u})$ and the Rubin estimator for $m \to \infty$ in the case of non-informative priors.

Theorem

For any response rate $f \in (0,1)$ and original sample size n

$$\lim_{m\to\infty} \mathrm{MSE}(\nu^2(\alpha_{*,u}, \beta_{*,u})) = \tfrac{2\sigma^4}{r^2}\left[\frac{1}{r-1} - \frac{(r-1)f}{n^2-3n-r+3}\right]. \tag{11}$$

and

$$\lim_{m\to\infty} \mathrm{MSE}(\nu^2_{Rub}) = \tfrac{2\sigma^4}{r-1}\left(\frac{1}{n} + \frac{1-f}{r}\right)^2. \tag{12}$$

Consequently,

$$\lim_{m\to\infty} \frac{\mathrm{MSE}(\nu^2(\alpha_{*,u},\beta_{*,u}))}{\mathrm{MSE}(\nu^2_{Rub})} = \frac{1}{r}\left[1 - \frac{(r-1)^2 f}{n^2-3n-r+3}\right]. \tag{13}$$

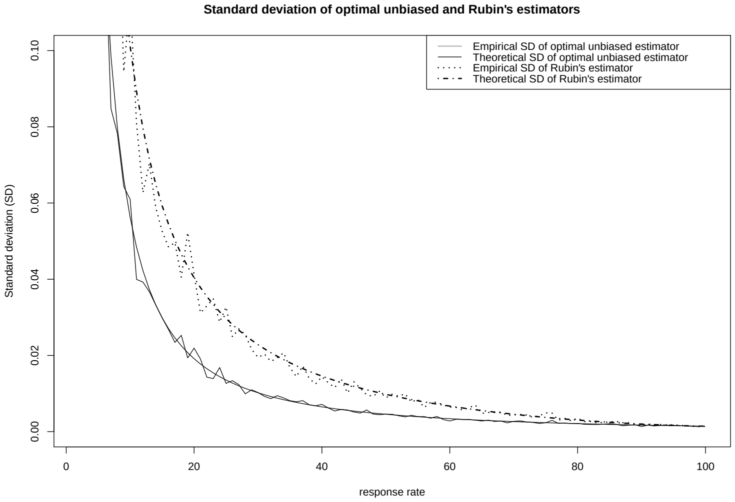

Figure: For non-informative prior Rubin's estimator is not optimal among unbiased estimators from the class $\mathfrak{R}$. The computations were done for $m = 5$, $n = 100$, $\sigma^2 = 1$, $\kappa = 1000$ and 100 repetitions for the empirical standard deviation (SD).

Recommendation

To estimate the variance/MSE of $\bar{X}_{\text{MImp}}$, instead of Rubin's estimator, use

$$\nu^2 = \frac{1}{f}\overline{U}_m - \frac{f}{n(1-f)}B_m$$

THANK YOU!!!

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