Modified acceptance sampling for statistical quality control

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Compounding parts of the statistical quality control

1. Management quality
2. Process quality charts (quality control and improvement)
3. Planned experiments (selection of the most important features)
4. Acceptance sampling (AS) (testing a production lot by a consumer). AS applied in
   - industry,
   - manufacturing,
   - receiving of agriculture production,
   - ...

Acceptance sampling in a war industry

Main terminology of AS formulated in 1925-26
by Dodge H. F. & H. G. Romig.

During the World War II, Dodge had an office in the Pentagon

- served as a consultant to the Secretary of War,
- was an instructor in more than 30 *quality control training conferences* for Army Ordnance.

The United States Military developed *sampling inspection schemes as part of the World War II effort*, MIL-STD-105E, valid until 1995.

Rheinmetall is an international integrated technology group that develops and sells components, systems and services for the security (munitions) and civil industries.
It uses statistical quality control in the ammunition manufacturing.

Acceptance sampling

E. G. Schilling developed the field further.

Production from a *producer* comes to a *consumer* in the lots. Objection – to accept a lot or to reject it (Schilling 2008). Possibilities:

- to accept without inspection,
- 100% inspection,
- acceptance sampling for *attributes* and *variables*.

Advantages of acceptance sampling:

- less expensive than 100% inspection,
- it is applicable in the case of *destructive* testing,
- fewer personal needed.

Disatvantages:

- the risk to accept "bad" lots and reject "good" ones,
- less information obtained about the production quality.

Types of the sampling plans

1. *Single* sampling plan. n items are selected from a lot at random. If # defectives d, $d \leq C \Rightarrow$ a lot is accepted;
   otherwise rejected.
2. *Double* sampling plan – 2 phase procedure.
   Let n_1 – size of the first sample.
   If # defectives d_1, $d_1 \leq C_1 \Rightarrow$ a lot is accepted.
   If # defectives $d_1 > C_2 > C_1 \Rightarrow$ a lot is rejected.
   If # defectives $C_1 < d_1 \leq C_2 \Rightarrow$ a second sample of size n_2 is drawn.
   Let d_2 # defectives observed. If $d_1 + d_2 \begin{cases} \leq C_2 & \Rightarrow \text{ lot accepted;} \\ > C_2 & \Rightarrow \text{ lot rejected.} \end{cases}$
3. *Multiple* sampling plan. An extension of double sampling plan to more than two phases.
4. *Sequential* sampling plan (Cochran's proposal)

Attribute sampling

Assumptions:

- items are *homogeneous* in a lot,
- they are selected to the sample *independently* with *equal probabilities*.

Let Y – random variable meaning # of defective items in a n-size sample from a N – size lot.

If lot size N is not taken into account or $n/N \sim 0 \Rightarrow$ then the distribution:

$$Y \sim \text{Hypergeometric} \Rightarrow \sim \text{Binomial}$$

Distribution function of Y

$$F(C \mid p) = \sum_{d=0}^{C} \binom{n}{d} p^d (1-p)^{n-d}, \quad C = 1, 2, \ldots$$

Single sampling for attributes – traditional approach

Notations. $\alpha,\ \beta \in (0,1)$; $0 < \alpha < \beta < 1$. $C = 0, 1, 2, ..., n$.
$p_0,\ p_1 \in (0,1)$; usually $p_0 < p_1$.
$1 - \alpha$ – probability to accept a lot with *allowable* proportion p_0 of defective items

$$F(C \mid p_0) = P(Y \le C \mid p_0) \ge 1 - \alpha, \tag{1}$$

β – probability to accept a lot with *unallowable* proportion p_1 of defective items:

$$F(C \mid p_1) = P(Y \le C \mid p_1) \le \beta \tag{2}$$

$\alpha = P(I\ type\ error)$ – **produce's risk** that a "good" lot is rejected.
$\beta = P(II\ type\ error)$ – **consumer's risk** that a "bad" lot is accepted.

Aim: to find the *smallest* sample size n satisfying both (1) and (2), in order to reach a compromise between producer and consumer.

Beta distribution

Why not to use **other sampling designs** to select the items from a lot?

Why not to study **distribution of the proportion** p of defective items in a lot?

Situation is similar to the audit sampling (Wywial, 2015).

Bayesian approach is used.

Beta distribution concentrated in the interval $(0, 1)$,

$$p \sim beta(a, b), \;\; a > 0, \;\; b > 0.$$

Its numerical characteristics:

$$E(p \mid a, b) = \frac{a}{a+b}; \quad Var(p \mid a, b) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Bayesian approach for posterior distribution of p

Let s – SRWR, n-size sample.
For any item $i \in s$; $I_i = 1$, if i is defective, $I_i = 0$ otherwise, $i = 1, ..., N$.
The sample data $y = \sum_{i \in s} I_i$.
Let $g(p) \sim \; beta(a, b)$, prior distribution for proportion p,
$f(y, p) = \binom{n}{d} p^y (1-p)^{n-y}$ – likelihood.
According to Bayesian theorem, posterior distribution for p:

$$g(p \mid y) = \frac{f(y \mid p) g(p)}{\int_0^1 f(y \mid p) g(p) \mathrm{d}p} \propto f(y \mid p) g(p).$$

Finally, the posterior distribution

$$g(p \mid y) \propto beta(a + y, b + n - y). \tag{3}$$

From here, a minimal sample size n satisfying (1), (2) will be found.
a, b are obtained assuming the admissible values of $E(p \mid a, b)$, $Var(p \mid a, b)$.

Notations for a stratified lot SRWR sampling design

Bayesian approach with SRWR sampling applied for each stratum separately.

Table: 1. Notations

Stratified lot	$\mathcal{U} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_H,\, \mathcal{U}_l \cap \mathcal{U}_h = \emptyset$
Lot strata sizes	$N = N_1 + \ldots + N_H$
Sample	$s = s_1 \cup \ldots \cup s_H,\ s_l \cap s_h = \emptyset$
Sample strata sizes	$n = n_1 + \ldots + n_H$
Proportions of defective items	$p_1, \ldots, p_H$
Likelihood functions	$f(y_1 \mid p_1), \ldots, f(y_H \mid p_H)$
Prior densities of the proportions	$g(p_1 \mid a_1, b_1), \ldots, g(p_H \mid a_H, b_H)$
Posterior densities of the proportions	$g(p_1 \mid y_1), \ldots, g(p_H \mid y_H)$

Stratified sampling design with proportional allocation of the sample size and SRWR

A large number J of the values is generated from the posterior distribution in each stratum:

$$p_1^{(j)}, \dots, p_H^{(j)}, \quad j = 1, \dots, J.$$

The simulated values of the strata posterior densities are aggregated:

$$p^{(j)} = \frac{N_1}{N} p_1^{(j)} + \dots + \frac{N_H}{N} p_H^{(j)}, \quad j = 1, \dots, J. \tag{4}$$

They are considered to be simulated values from the posterior density of p.

These values are used for the statistical inference from the posterior.

Unequal probability sampling with replacement and with probabilities proportional to size

x – auxiliary size variable. Selection probabilities
$q_k = x_k / \sum_{i \in \mathcal{U}} x_i$, $k = 1, ..., N$.
s – n-size SRWR *pps*-sample from N-size population.
Inclusion probabilities $\pi_k = 1 - (1 - q_k)^n$, $k \in \mathcal{U}$.
$I_k = 0$ or $I_k = 1$, $k = 1, ..., N$.

Task: to find a posterior distribution of a proportion $p = \sum_{k=1}^{N} I_k / N$.

A discrete study variable I is replaced by a continuous latent variable z (Little, 2022) satisfying a model

$$z_k = p + \sqrt{\pi_k} \cdot \varepsilon_k, \quad \varepsilon_k \sim \mathcal{N}(0, \sigma^2); \quad k \in s, \ \sigma > 0.$$

This heterogeneous model is rearranged to a homogeneous model:

$$\frac{z_k}{\sqrt{\pi_k}} = \frac{p}{\sqrt{\pi_k}} + \varepsilon_k, \quad \varepsilon_k \sim \mathcal{N}(0, \sigma^2); \quad k \in s.$$

Continuation 1

Change of notations:

$$z_k^* = z_k/\sqrt{\pi_k}, \quad \mu_k^* = p/\sqrt{\pi_k} \tag{5}$$

The model becomes $\quad y_k^* = \mu_k^* + \varepsilon_k,\ k \in s$

Replacement $\quad \mu_k^*$ by $\tilde{\mu}$ for $k \in s$;

$$z_k^* = \tilde{\mu} + \varepsilon_k,\ k \in s \ \Rightarrow$$

data z^* distribution $\mathcal{N}(\tilde{\mu}, \sigma^2)$ with σ^2 unknown.

Posterior marginal distribution of $\tilde{\mu}$:

$$g(\tilde{\mu} \mid z_k^*, k \in s, \sigma^2) \propto f(z_k^*, \in s, \mid \tilde{\mu}, \sigma^2) g(\tilde{\mu} \mid \sigma^2) g(\sigma^2).$$

It is obtained from likelihood: $\quad f(z_k^*, k \in s \mid \tilde{\mu}, \sigma^2) \sim \mathcal{N}(\tilde{\mu}, \sigma^2).$

Prior distributions ($\kappa_n = n + 1$, σ_0^2 guessed):

$$\begin{aligned} g(\tilde{\mu} \mid \sigma^2, z_k^*, k \in s) &\sim \mathcal{N}(\tilde{\mu}_0, \sigma^2/\kappa_n), \\ g(\sigma^2) &\sim Invgamma\Big(\frac{1}{2}, \frac{\sigma_0^2}{2}\Big) \end{aligned}$$

Continuation 2

Posterior marginal density for $\tilde{\mu}$:

$$g(\tilde{\mu} \mid z_k, k \in s; \sigma^2) \sim \mathcal{N}(\tilde{\mu}_n, \sigma^2/\kappa_n), \quad \tilde{\mu}_n = \frac{1}{\kappa_n}\mu_0 + \frac{y}{\kappa_n},$$

$\tilde{\mu}$ is multiplied by $\sqrt{\pi_k}$:

$$g(\tilde{\mu}\sqrt{\pi_k} \mid \sigma^2,\ z_k^*,\ k \in s) \sim \mathcal{N}\big(\sqrt{\pi_k}\tilde{\mu}_n, (\sqrt{\pi_k}\sigma)^2/\kappa_n\big), \quad k \in s. \tag{6}$$

According to (5), we have $\tilde{\mu}\sqrt{\pi_k} = p\tilde{\mu}/\mu_k^*, \ \ k \in s$ (because $\mu_k^* = p/\sqrt{\pi_k}$).

The sample average of these values

$$\frac{1}{n}\sum_{k \in s} \tilde{\mu}\sqrt{\pi_k} = p\tilde{\mu}\frac{1}{n}\sum_{k \in s}\frac{1}{\mu_k^*} = p\frac{\tilde{\mu}}{H_n} \sim p,$$

where H_n is a harmonic mean of the expressions μ_k^*, $k \in s$.

Hypothesis formulation

$$\mathcal{P}_0 = (0, c] \quad \text{and} \quad \mathcal{P}_1 = (c, 1), \quad c \in (0, 1). \tag{7}$$

The hypothesis H_0 is tested against the alternative hypothesis H_1:

$$H_0: \; p \in \mathcal{P}_0, \quad H_1: \; p \in \mathcal{P}_1. \tag{8}$$

These hypotheses are mutually exclusive and exhaustive.

Introduction of the loss function $L(p \in \mathcal{P}_i, H_j)$, which means the loss or the risk of accepting the hypothesis H_j when $p \in \mathcal{P}_i$ for $i, j = 1, 2$.
There are four possible cases:

$$\begin{aligned} L(p \in \mathcal{P}_i, H_i) &= 0 \quad \text{for} \quad i = 1, 2; \\ L(p \in \mathcal{P}_0, H_1) &= \alpha \quad \text{for} \quad \alpha \in (0, 1); \\ L(p \in \mathcal{P}_1, H_0) &= \beta \quad \text{for} \quad \beta \in (0, 1). \end{aligned}$$

Odds ratio

The expected posterior loss under acceptance of each of the hypotheses taking into account the risk of the producer and that of the consumer:

$$\begin{aligned} E\big(L(H_0)\big) &= \beta \cdot P(H_1 \mid y); \\ E\big(L(H_1)\big) &= \alpha \cdot P(H_0 \mid y). \end{aligned}$$

In Bayesian hypotheses testing, the decision is taken in favour of the hypothesis H_0 with the minimal posterior loss (Koch, 2007). Therefore, H_0 is not rejected for

$$\frac{E(L(H_1))}{E(L(H_0))} > 1 \quad \text{or} \quad \frac{P(H_0 \mid y)}{P(H_1 \mid y)} > \frac{\beta}{\alpha}. \tag{9}$$

Probability $P(H_0 \mid y) = P(p \leq c \mid y)$ is a value of the posterior distribution function of the proportion p at the point $c \in (0, 1)$. The last inequality becomes

$$OR(y) = \frac{P(p \leq c \mid y)}{1 - P(p \leq c \mid y)} > \frac{\beta}{\alpha}. \tag{10}$$

If (10) is satisfied, there is no reason to reject the null hypothesis H_0 in (8).

Hypothesis testing by odds ratio

The odds ratio of the posterior distribution:
$OR(y) = P(p \le c \mid y)/P(p > c \mid y)$; $K = \beta/\alpha$.

Table: 2. The Evidence Categories for H_0 and H_1

$OR(y)$	Interpretation
$> 10K$	Strong evidence for hypothesis H_0
$3K \div 10K$	Moderate evidence for hypothesis H_0
$2K \div 3K$	Week evidence for hypothesis H_0
$K \div 2K$	Insignificant evidence for hypothesis H_0
K	No evidence
$K/2 \div K$	Insignificant evidence for alternative H_1
$K/3 \div K/2$	Weak evidence for alternative H_1
$K/10 \div K/3$	Moderate evidence for alternative H_1
$< K/10$	Strong evidence for alternative H_1

$OR(y)$ classification table

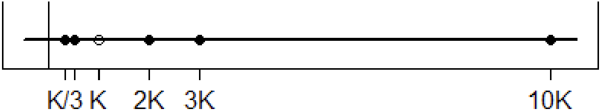

Table: 3. Notations for a simulation study

$OR(y)$ categories	Moderate, strong evidence for H_0 $OR(y) > 3K$	Weak evidence for H_0 $OR(y) \in [2K; 3K]$	Insignificant evidence for H_0 $OR(y) \in (K; 2K)$
Frequency	$m_{03}(n)$	$m_{02}(n)$	$m_{01}(n)$
Rel. freq.	$f_{03}(n)$	$f_{02}(n)$	$f_{01}(n)$
$OR(y)$ categories	Insignificant evidence for H_1 $OR \in (K/2; K)$	Weak evidence for H_1 $OR(y) \in [K/3; K/2]$	Moderate, strong evidence for H_1 $K/3 > OR(y)$
Frequency	$m_{11}(n)$	$m_{12}(n)$	$m_{13}(n)$
Rel. freq.	$f_{11}(n)$	$f_{12}(n)$	$f_{13}(n)$

Simulation study

p_0 – admissible proportion of defective items; p_1 – alternative proportion. Producer's risk $\alpha = 0.05$, the consumer's risk is $\beta = 0.1$, the constant $K = \beta/\alpha = 2$.

The threshold $c = 0.08$, the highest admissible proportion of defective items. Population simulated:

- $\mathcal{U} \sim Bern(N, p_1)$, $0 < p_0 < p_1$,
- prior density is reflecting p_0,
- large number M of n-size samples drawn. For each sample, the posterior density obtained
  - explicitly or
  - J samples drawn from it.

Notations for the posterior distribution odds ratios $OR(y)$ classification according to Table 2 are presented in Table 3.

The class frequencies $m_{ij}(n)$. The relative class frequencies $f_{ij}(n) = m_{ij}(n)/M$, $i = 0, 1$, $j = 1, 2, 3$, are included in the tables 4, 5.

Simple random sampling with replacement

The lot $\mathcal{U}$ values are simulated with $Bern(N, p_1)$, $p_1 = 0.06, N = 10\,000$; $\widehat{p}_1 = 0.0583$.

Table: 4. Choice of the sample size for a SRSWR sampling, $p_0 = 0.04$, $M = 10\,000$

Sample size n	20	30	40	50	100	150
$f_{03}(n)$	**0.8921**	0.9091	0.9150	0.9293	0.8717	0.9037
$f_{02}(n)$	0.0825	0.0654	0.0561	0.0000	0.0632	0.0432
$f_{01}(n)$	0.0208	0.0191	0.0221	0.0621	0.0530	0.0392
$f_{11}(n)$	0.0046	0.0054	0.0052	0.0067	0.0070	0.0107
$f_{12}(n)$	0.0000	0.0005	0.0011	0.0013	0.0045	0.0021
$f_{13}(n)$	0.0000	0.0005	0.0005	0.0006	0.0006	0.0011
$OR(y) > K$	0.9954	0.9936	0.9932	0.9914	0.9879	0.9861
$OR(y) < K$	0.0046	0.0064	0.0068	0.0086	0.0121	0.0139

Stratified simple random sampling with replacement

Table: 5. Choice of the sample size for a stratified SRWR sampling, $p_0 = 0.0408$, $p_1 = 0.1333$, $\widehat{p}_1 = 0.1408$, $N = 10\,000$, $M = 10\,000$

Sample size n	100	150	200	250	300
$f_{03}(n)$	0.1962	0.0585	0.0175	0.0045	0.0006
$f_{02}(n)$	0.0924	0.0382	0.0136	0.0053	0.0015
$f_{01}(n)$	0.1999	0.1163	0.0527	0.0213	0.0069
$f_{11}(n)$	0.2126	0.1803	0.1013	0.0538	0.0225
$f_{12}(n)$	0.1026	0.1182	0.0874	0.0557	0.0286
$f_{13}(n)$	0.1948	0.4874	0.7268	**0.8591**	0.9398
$OR(y) > K$	0.4885	0.2130	0.0838	0.0311	0.0090
$OR(y) < K$	0.5100	0.7859	0.9155	0.9186	0.9909

Summary of the simulation results

1. The lots with the probability p_1 of the item with the defectiveness $p_1 << c$ are classified in Table 4 in favour of H_0. The relative frequencies $f_{03}(n)$ are increasing with an increasing sample size, the suitable minimal sample size for hypothesis (8) testing can be found.
2. The lots with the probability p_1 of a defective item $p_1 \approx c$ show different results. A high number of $OR(y)$ is inconclusive (close to K): relative frequencies do not depend on the sample size for all classes. Some of the lots with probability of defective items close to the threshold level c are successfully tested in favour of H_0.
3. For the lots with high probability p_1 of the item being defective, $p_1 >> c$, as in Table 5, the null hypothesis in (8) may be rejected and an alternative H_1 not rejected. It means the lots are terribly defective. The smallest sample size for decision H_1 may be found.

Conclusions and other approaches

Conclusions

1. Items in a lot received by the consumer should not be necessary homogeneous.
2. Distribution for a proportion of defective items can be evaluated for any probability sampling design for which posterior distribution of p is available, and sample size needed can be found.

Ohter approaches

- Application for real data.
- Penalized spline probit model for p may be helpful
- Asymmetric Wilson confidence interval for a proportion of defectives can be estimated and sample size needed to get the upper bound of the interval estimated (Valliant et al. 2018).
- Application of unequal probability sampling design for accelerated life time testing under various life time distributions for sample size finding.

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Thank you!