A New Extension of Exponentiated Standard Logistic Distribution: Properties and Applications

P. Sulewski, M. Alizadeh, P.J.Hazarika, J. Das, G.G. Hamedani

Pomeranian University in Slupsk, Poland; Persian Gulf University, Bushehr, Iran; Dibrugarh University, Dibrugarh, India; Dibrugarh University, Dibrugarh, India; Marquette University, Wisconsin, U.S.A

IV Kongres Statystyki Polskiej, 02-04.07.2024

Advertisement

PSGoft (CRAN) - modifiied Lilliefors GoFT for normality

PSDistr (CRAN) - six distributions derived from normal distribution

PSIndependenceTest (CRAN) - two independence tests for two-way, three-way and four-way contingency tables

1 Introduction

2 Main properties

3 Estimation

4 Applications

5 Appendix

The logistic (L) distribution, created but not named by Poisson in 1824, is an important continuous probability distribution. Its cdf is the L function, which appears in L regression. The L function was probably first invented in 1838 to describe population growth by the Belgian mathematician Verhulst (1838), who gave it its name Verhulst (1845).

The L distribution resembles the normal distribution in shape, but has heavier tails (higher kurtosis). The logistic distribution is a special case of the Tukey lambda distribution.

Many generalized types of the L distribution have been proposed. These generalizations developed to extend the scope of the L model to asymmetric probability curves and improve fit in non-central probability regions include: the type-I, type-II, type-III generalized L(GL),type-IV GL, five-parameter GL, skew L, extended type-I, extended type-II, extended type-III, extension of the skew L, new skew L, beta GL, modified skew L, skew type-I GL, five parameter type-II GL and exponentiated exponential.

Omukami (2022) introduced a new class of probability distribution known as exponentiated standard L (ESL) distribution with cdf and pdf given by, respectively

$$F(x; a) = (1 + \exp(-x))^{-a}, \tag{1}$$

$$f(x; a) = \frac{a \exp(-x)}{(1 + \exp(-x))^{a+1}}. \tag{2}$$

According to Burr system (Burr 1942) this distribution is called Burr II distribution.
Despite numerous new distributions being introduced in the literature, showcasing their versatility, there has been limited investigation into distributions accommodating multiple modes.
Alpha skew normal, alpha skew logistic, alpha skew Laplace, generalized alpha skew normal etc. are some of the probability distributions which exhibit at most two modes.
Besides, several probability distributions were developed to accommodate data with up to three modes. Examples include the alpha beta skew logistic, alpha beta skew normal, generalized alpha beta skew normal and alpha beta skew Laplace, among others.

Moreover, under Balakrishnan Mechanism Balakrishnan (1988), some novel family of probability distribution were reported in the literature which also accommodate more than single modes. Balakrishnan alpha skew normal, Balakrishnan alpha skew logistic, Balakrishnan alpha skew laplace, Balakrishnan alpha skew generalized t, generalized Balakrishnan alpha skew norma were some models showing flexibility for fitting data with uni-bimodality behavior.
On the other hand, models like Balakrishnan alpha beta skew normal and Balakrishnan alpha beta skew Laplace allow fitting data up to three modes. Besides these, log alpha skew normal and log Balakrishnan alpha skew normal were two examples of new distributions considering positive support and allowing data to fit with uni-bimodality character.

Additionally, multimodal skew normal, multimodal skew laplace, multimodal alpha skew normal, multimodal Balakrishnan alpha skew normal were some other families of distribution reported in the literature to present data with multimodality.
In Martinez (2022) also several symmetric and asymmetric families of probability distributions were suggested for fitting data with multiple modes. Among these models were the trimodal skew normal, flexible alpha normal (FAN), exponentiated FAN, and others. Following this research work, Pathak (2023) introduced trimodal skew logistic distribution for fitting data with at most three modes.
Das (2023) also introduced an asymmetric version of the flexible alpha normal (FAN) which is known as flexible alpha skew normal distribution.

While numerous studies have focused on uni-bimodal symmetric or asymmetric probability distributions, there remains ample opportunity to explore novel families of distributions capable of accommodating data with multiple modes. This article addresses a new extension of the exponentiated standard logistic distribution (Omukami 2022), which demonstrates flexibility in fitting data with up to two modes.

The presentation discusses important mathematical properties of the new distribution and provides graphical visualizations.

Furthermore, the adaptability and utility of the new distribution are assessed using real-life datasets.

PDF

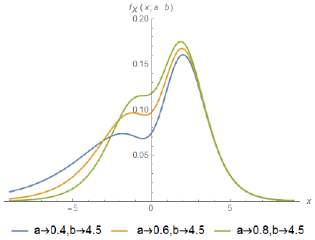
If X is a random variable then X is said to follow an Extended Exponentiated Standard Logistic Distribution if its probability density function is given by

$$f(x) = \frac{e^{-x}(1+e^{-x})^{-a-1}\left[a+(b-a)(1+e^{-x})^{-b}\right]}{[(1+e^{-x})^{-a}+1-(1+e^{-x})^{-b}]^2} \tag{3}$$

where, $x \in R$ and $a, b > 0$ are two shape parameters. It denoted as $EESL(a, b)$.

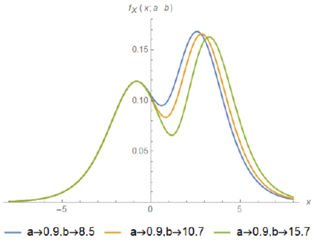
Special Cases

i. If $a = b$, then $X \sim$ Exponentiated Standard Logistic distribution (Omukami,2022).
ii. If $a = b = 1$, then $X \sim$ Standard Logistic distribution.



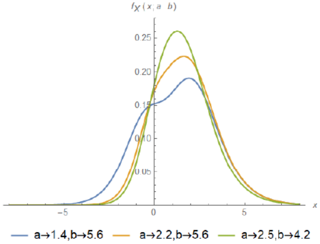
fX (x;a b)
0.20
0.15
0.10
0.05
−5
0
5
x
a→0.4,b→4.5
a→0.6,b→4.5
a→0.8,b→4.5

(a)



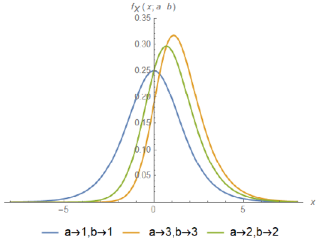
fX (x;a b)
0.15
0.10
0.05
−5
0
5
x
a→0.9,b→8.5
a→0.9,b→10.7
a→0.9,b→15.7

(b)



fX (x;a b)
0.25
0.20
0.15
0.10
0.05
−5
0
5
x
a→1.4,b→5.6
a→2.2,b→5.6
a→2.5,b→4.2

(c)



fX (x;a b)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
−5
0
5
x
a→1,b→1
a→3,b→3
a→2,b→2

(d)

The $EESL(a, b)$ can be used to departure from the Gaussian distribution with PDF $\phi(x; m, s)$. The similarity measure between our proposal and the $N(0, s)$ is given by (Sulewski 2020)

$$M(a, b, s) = \int_{-\infty}^{\infty} min\Big[f(x; a, b), \Phi(x; 0, s)\Big]. \tag{4}$$

Numerical examination in Mathcad gives $M_{max}(0.994, 1.017, 1.638) = 0.967$, so the $EESL(0.994, 1.017)$ is the $N(0, 1.638)$ in 96.7%. Exemplary $M(a, b, s)$ values are presented in Table 1.

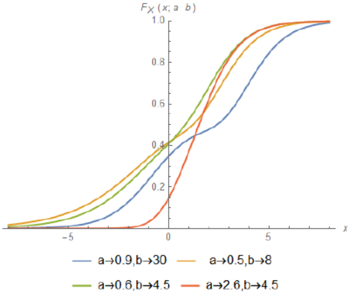
Table: Similarity measure $M(a, b, s)$ between the $EESL(a, b)$ and $N(0, 1.638)$.

I	0.994	0.887	0.719	0.581	0.462	0.359	0.269	0.190	0.122	0.061
II	1.017	0.809	0.607	0.461	0.352	0.269	0.205	0.155	0.117	0.087
$M(a, b, 1.638)$	0.967	0.950	0.900	0.850	0.800	0.750	0.700	0.650	0.600	0.550

CDF

The Corresponding Cumulative distribution function (CDF) of the *EESL*(a, b) probability distribution defined in (3) is given by

$$F(x) = \frac{(1+e^{-x})^{-a}}{(1+e^{-x})^{-a} + 1 - (1+e^{-x})^{-b}} \tag{5}$$



$$E(X) = \frac{1}{F(y)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} w_{i,j} \left({}_3F_2(1,1,p;2,2;-1) - \frac{{}_2F_1(p-1,p-1;p;-1)}{(p-1)^2} \right)$$

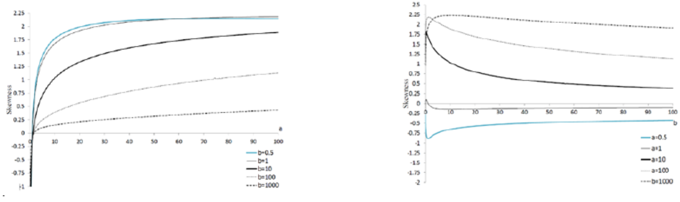
$$E(X^2) = \frac{1}{F(y)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} w_{i,j} \left(2 \left(\frac{{}_3F_2(p-1,p-1,p-1;p,p;-1)}{(p-1)^3} + {}_4F_3(1,1,1,p;2,2,2;-1) \right) \right)$$

$$E(X^3) = \frac{6}{F(y)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} w_{i,j} \Bigg({}_5F_4(1,1,1,1,p;2,2,2,2;-1) - \frac{6\, {}_4F_3(p-1,p-1,p-1,p-1;p,p,p;-1)}{(p-1)^4} \Bigg)$$

$$E(X^4) = \frac{1}{F(y)} \sum_{i=0}^{\infty} \sum_{j=0}^{i} w_{i,j} \Bigg(24 \Bigg(\frac{{}_5F_4(p-1,p-1,p-1,p-1,p-1;p,p,p,p;-1)}{(p-1)^5} + {}_6F_5(1,1,1,1,1,p;2,2,2,2,2;-1) \Bigg) \Bigg).$$

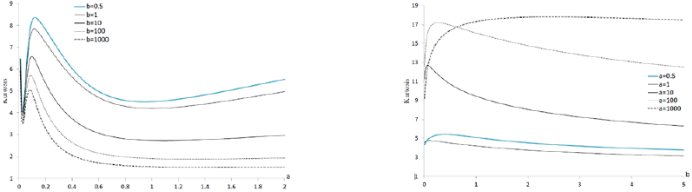
where, ${}_qF_p(a;b;z)$ is the generalized hypergeometric function (Virchenko et al., 2001)

Skewness and kurtosis



(a) (b)

Figure 3.: Skewness of $EESL(a, b)$ distribution.

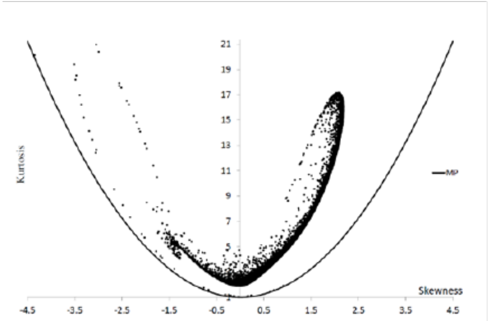


(a) (b)

Figure 4.: Kurtosis of $EESL(a, b)$ distribution .

Skewness and kurtosis

We calculate γ_1 and γ_2 for 10^5 randomly values of $a = Unif(0, 100)$ and $b = Unif(0, 100)$. Figure 5 presents a set of points (γ_1, γ_2) located in a rectangle $(-4.5, 4.5) \times (1, 21.25)$. Symbol MP denotes the Malakhov parabola $\gamma_2 = \gamma_1^2 + 1$. We obtain $\gamma_1 \in (-4.351, 2.183), \gamma_2 \in (1.026, 20.979)$.



Bowley's (B) and Moors' (M) measure

The analysis of γ_1 and γ_2 can be investigated using x_p quantiles, especially when moments do not exist. Quantile alternatives for γ_1 and γ_2 are B and M measures, respectively, defined as

$$B = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}, \quad M = \frac{x_{0.875 - x_{0.625} + x_{0.375} - x_{0.125}}}{x_{0.75} - x_{0.25}} \tag{6}$$

However, it should be noted that the values of quantile alternative measures differ significantly from classic ones, e.g. for $a = 0.01, b = 1$ we obtain $\gamma_1 = 0.852, B = -0.926$ and $\gamma_2 = 1.797, M = 1.318$.

Quantiles and pseudo-random number generator

Theorem Let, $X \in EESL(a, b)$. Then the $p-th$ $(0 < p < 1)$ quantile is a solution of the equation

$$(1+\exp(-x_p))^{-a}(1-p) - p\left[1-(1+\exp(-x_p))^{-b}\right] = 0 \qquad (7)$$

Theorem Let R $Unif(0,1)$. The pseudo-random number generator of X is a solution of the equation

$$R\left[(1+\exp(-X))^{-a} + 1 - (1+\exp(-X))^{-b}\right] = 0 \qquad (8)$$

Moments of order statistics

Let $X_{i,n}$ be the i-th order statistic $(X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n})$ in a sample of size n from the NEESL(a, b) and $u(x) = 1 + e^{-x}$. The k-th moment of the i-th order statistic $X_{i,n}$ is defined as

$$\alpha_{k,i,n} = E\left(X_{i,n}^{k}\right) = \int_{-\infty}^{\infty} x^k f_{i,n}(x; a, b) dx,$$

where

$$f_{i,n}(x; a, b) = \binom{n}{i-1} \frac{(n-i+1)[u(x)-1]\left[a+(b-a)u(x)^{-b}\right]\left[1-u(x)^{-b}\right]^{n-i}}{u(x)^{ai+1}\left[u(x)^{-a}-u(x)^{-b}+1\right]^{n+1}}.$$

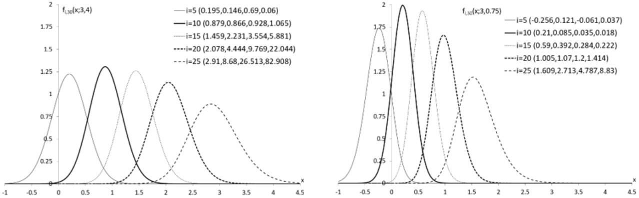


Figure 4. PDF of the $X_{5i,30}$ $(i = 1,2, \ldots,5)$ of the $BPL(0,1,c)$ distribution

Estimation

$$f(y;\mu,\sigma,a,b)=\frac{\exp\left(-\frac{x-\mu}{\sigma}\right)\left(1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-a-1}\left[a+(b-a)\left(1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-b}\right]}{\sigma\left[\left(1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-a}+1-\left(1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-b}\right]^{2}} \quad (14)$$

$$F(y;\mu,\sigma,a,b)=\frac{\left(1+\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{-a}}{\left(1+\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{-a}+1-\left(1+\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{-b}} \quad (15)$$

Looking through the literature devoted to the distribution theory, we find that the most dominant method of parameter estimation is the maximum likelihood (ML) method. However, the younger the paper, the more often the parameters are estimated also using other methods, e.g. the ordinary least-squares (OLS).

Estimation

A very interesting proposal is using natural estimation measure, i.e. the least absolute values (LAW). The LAW measures the absolute values of the differences between the empirical and theoretical CDFs. To obtain the OLS and LAW estimates of the NEELS parameters, we minimize the following objective functions, respectively

$$OLS = \sum_{i=1}^{n} \left[\frac{i}{n+1} - F(x_i; \mu, \sigma, a, b) \right]^2, \tag{9}$$

$$LAW = \sum_{i=1}^{n} \left| \frac{i}{n+1} - F(x_i; \mu, \sigma, a, b) \right|. \tag{10}$$

Estimation

ML, OLS and LAW methods effectively estimate the model parameters. The results also indicate that when sample size increases, the bias and MSE of the estimates decrease, indicating the asymptotic consistency of the estimates. Moreover, the lowest absolute average of bias and the lowest absolute average of MSE occur for the scale parameter. MSE of b parameter for analyzed method increases with the parameter values. The lowest average bias and the lowest average MSE are for the ML method.

Applications

This section represents the applicability and adaptability of the proposed distribution in comparison to other ten models such as the Logistic (L), Extended L(EL), Type-I Generalized L (IGL), Type-II GL (IIGL), Type-III GL (IIIGL), Skew L (SL), Alpha Skew L (ASL), Alpha beta skew L (ABSL), Modified SL (MSL) and Exponentiated Exponential L (EEL).
The EESL model is compared to the above models using visual techniques (see Figure) and numerical measures. The Akaike (AIC), Bayes (BIC), Hannan-Quinn (HQIC) information criteria (IC) and Kolmogorov-Smirnov statistics (KS) are calculated for the ML method, while KS and objective functions (17, 18) are calculated for the methods OLS and LAW. Recall that the AIC, BIC and HQIC are defined as

$$AIC = -2l + 2p,\ BIC = -2l + p\ln(n),\ HQIC = -2l + 2p\ln(\ln(n))$$

Applications

All calculations were performed in the R program in two ways. The first way uses the "fitdistr" and "optim" functions, and the second way uses the "GenSA" function. To avoid local maxima (ML) and minima (OLS, LAW), the optimization procedure was performed 10^2 times with random values of the initial parameters, which are widely scattered in the parameter space. Because we used two ways to obtain parameter estimates, the final parameter estimates were the better maximizing (15) or minimizing (17, 18) values.

Example 1

The Environmental Performance Index (EPI) dataset with $n = 163$ observations (Chakraborty 2014).

Table 4.: MLE, I, IC and KS values of models (M) fitted to 163 EPI observations

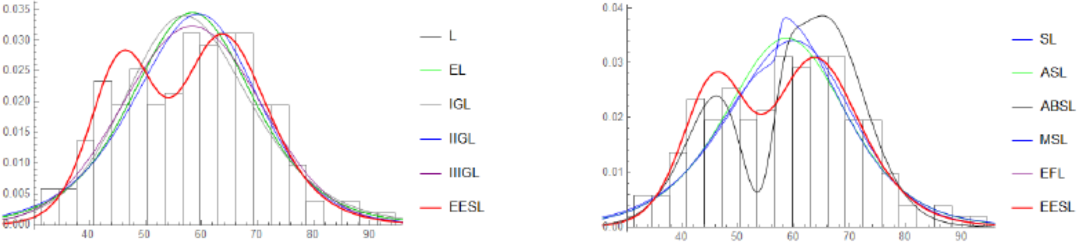
M	μ	σ	α	β	γ	a	b	l	AIC	BIC	HQIC	KS
L	58.411	7.254						-645.165	1294.330	1300.518	1296.842	0.075
EL	55.216	7.254	0.644					-645.165	1296.330	1305.611	1300.098	0.075
IGL	53.902	8.154	1.457					-644.913	1295.827	1305.108	1299.595	0.068
IIGL	61.959	7.946				1.357		-644.813	1295.626	1304.908	1299.395	0.078
IIIGL	58.369	138.298				250.204		-641.297	1288.594	1297.875	1292.362	0.063
SL	63.482	7.765	-0.508					-644.652	1295.305	1304.586	1299.073	0.078
ASL	59.185	7.242	0.034					-645.150	1296.299	1305.581	1300.067	0.076
ABSL	57.669	2.056	-0.457	-0.053				-642.589	1293.178	1305.553	1298.202	0.100
MSL	57.269	7.290	21.912		-0.226			-644.532	1297.064	1309.439	1302.088	0.081
EEL	195.493	37.673				22.835	136.515	-641.348	1290.693	1303.068	1295.717	0.063
NEESL	46.064	5.616				1.927	15.681	**-636.338**	**1280.676**	**1293.051**	**1285.700**	**0.049**

Example 1

Table 5. OLS, LAW and KS values of models (M) fitted to 163 EPI observations

M	OLS	KS	LAW	KS
L	0.095	0.057	3.369	0.058
EL	0.095	0.057	3.369	0.058
IGL	0.080	0.061	2.652	0.076
IIGL	0.069	0.059	2.434	0.067
IIIGL	0.075	0.051	2.983	0.052
SL	0.070	0.059	2.693	0.062
ASL	0.079	0.060	3.131	0.058
ABSL	0.058	0.056	2.359	0.068
MSL	0.072	0.065	3.042	0.058
EEL	0.059	0.056	2.374	0.063
NEESL	**0.020**	**0.035**	**1.449**	**0.037**

Example 1



0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
40
50
60
70
80
90
L
EL
IGL
IIGL
IIIGL
EESL
0.04
0.03
0.02
0.01
0.00
40
50
60
70
80
90
SL
ASL
ABSL
MSL
EFL
EESL

(a) (b)

Example 2

the measurements of gauge lengths data set reported by Kundu and Raqab 2009.

Table 7. MLE, I, IC and KS values of models (M) fitted to measurements of gauge lengths

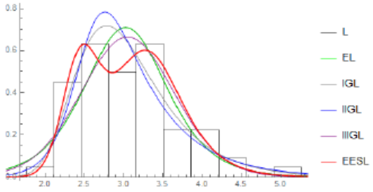
M	μ	σ	α	β	γ	a	b	l	AIC	BIC	HQIC	KS
L	3.024	0.352						-59.329	122.659	126.945	124.345	0.094
EL	2.906	0.352	0.715					-59.330	124.659	131.089	127.188	0.095
IGL	1.023	0.504	32.817					-56.509	119.018	**125.447**	121.546	0.088
IIGL	2.538	0.200				0.330		-57.957	121.913	128.343	124.442	0.108
IIIGL	3.048	0.931				5.052		-58.797	123.595	130.024	126.124	0.097
SL	2.328	0.550	3.713					-56.794	119.589	126.018	122.117	0.990
ASL	2.812	0.338	-0.209					-58.572	123.143	129.572	125.672	0.090
ABSL	3.303	0.113	-0.133	0.047				-58.014	124.029	132.601	127.400	0.131
MSL	2.333	0.547	-3.776		130.109			-57.014	122.029	130.601	125.400	0.101
EEL	1.055	1.184				108.219	2.804	-56.429	120.858	129.431	124.230	0.083
NEESL	2.148	0.308				5.027	28.794	**-55.018**	**118.036**	126.608	**121.407**	**0.061**

Example 2

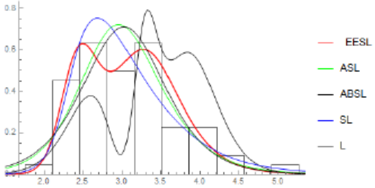
Table 8. OLS, LAW and KS values of models (M) fitted to measurements of gauge lengths

M	OLS	KS	LAW	KS
L	0.059	0.079	1.498	0.078
EL	0.059	0.079	1.498	0.078
IGL	0.046	0.064	1.328	0.074
IIGL	0.053	0.069	1.418	0.076
IIIGL	0.053	0.072	1.470	0.069
SL	0.048	0.066	1.341	0.075
ASL	0.054	0.070	1.414	0.076
ABSL	0.054	0.070	1.413	0.076
MSL	0.048	0.066	1.345	0.075
EEL	0.043	0.063	1.257	0.073
NEESL	**0.021**	**0.050**	**0.964**	**0.059**

Example 2



(a)



(b)

Figure 8.: Observed and expected densities for measurements of gauge lengths.

Appendix A

Let $\mu, \alpha, \beta \in R;\ \sigma, a, b > 0; \gamma \geq -1$. The pdfs of the models used in Section 7 are:

$$f_L(x;\mu,\sigma) = \frac{exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2},\ f_{EL}(x;\mu,\sigma,\alpha) = \frac{\alpha exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left[\alpha+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2},$$

$$f_{IGL}(x;\mu,\sigma,\alpha) = \frac{\alpha exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{\alpha+1}},\ f_{IIGL}(x;\mu,\sigma,a) = \frac{\delta exp\left(-a\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{a+1}}.$$

$$f_{IIIGL}(x;\mu,\sigma,a) = \frac{exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma B(a,a)\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{2a}},\ f_{SL}(x;\mu,\sigma,\alpha) = \frac{2exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(-\alpha\frac{x-\mu}{\sigma}\right)\right]\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2}.$$

$$f_{ASL}(x;\mu,\sigma,\alpha) = \frac{\left[3\left(1-\alpha\frac{x-\mu}{\sigma}\right)^2+3\right]exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma(6+\alpha^2\pi^2)\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2},$$

$$f_{ABSL}(x;\mu,\sigma,\alpha,\beta) = \frac{\left[\left(1-\alpha\frac{x-\mu}{\sigma}-\beta\left(\frac{x-\mu^3}{\sigma}\right)\right)^2+1\right]exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left(2+\frac{\alpha^2\pi^2}{3}+\frac{31\beta^2\pi^6}{21}+\frac{14\alpha\beta\pi^4}{15}\right)\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2},$$

$$f_{MSL}(x;\mu,\sigma,\alpha,\gamma) = \frac{2exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma(\gamma+2)\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^2}\left[1+\frac{\gamma exp\left(-\alpha\frac{x-\mu}{\sigma}\right)}{1+exp\left(-\alpha\frac{x-\mu}{\sigma}\right)}\right],$$

$$f_{EEL}(x;\mu,\sigma,a,b) = \frac{abexp\left(-b\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{b+1}}\left[1-\frac{exp\left(-b\frac{x-\mu}{\sigma}\right)}{\left[1+exp\left(-\frac{x-\mu}{\sigma}\right)\right]^b}\right]^{\delta-1}.$$

Appendix B

Appendix B

Environmental Performance Index dataset: 32.1, 33.3, 33.7, 36.3, 36.4, 37.6, 38.4, 39.4, 39.5, 39.6, 40.2, 40.7, 40.8, 41.0, 41.3, 41.7, 41.8, 41.9, 42.0, 42.3, 42.3, 42.8, 43.1, 43.9, 44.0, 44.3, 44.3, 44.4, 44.6, 44.6, 44.6, 44.7, 45.9, 47.0, 47.1, 47.3, 47.8, 47.9, 48.0, 48.3, 48.3, 48.9, 49.0, 49.2, 49.8, 49.9, 50.1, 50.3, 50.8, 51.1, 51.1, 51.2, 51.3, 51.3, 51.3, 51.4, 51.4, 51.6, 54.0, 54.0, 54.2, 54.3, 54.4, 54.6, 55.3, 55.9, 56.1, 56.3, 56.4, 57.0, 57.1, 57.3, 57.3, 57.9, 58.0, 58.1, 58.2, 58.8, 59.0, 59.1, 59.1, 59.2, 59.3, 59.6, 59.7, 60.0, 60.4, 60.4, 60.5, 60.6, 60.6, 60.8, 60.9, 61.0, 61.2, 62.0, 62.2, 62.4, 62.5, 62.9, 63.1, 63.4, 63.5, 63.5, 63.6, 63.7, 63.8, 64.6, 65.0, 65.0, 65.4, 65.6, 65.7, 65.7, 65.9, 65.9, 66.4, 66.4, 67.0, 67.1, 67.3, 67.4, 67.8, 68.0, 68.2, 68.2, 68.3, 68.4, 68.7, 69.1, 69.1, 69.2, 69.3, 69.3, 69.4, 69.6, 69.8, 69.9, 70.6, 71.4, 71.4, 71.6, 72.5, 72.5, 73.0, 73.1, 73.2, 73.3, 73.4, 74.2, 74.5, 74.7, 76.3, 76.8, 78.1, 78.1, 78.2, 80.6, 81.1, 86.0, 86.4, 89.1, 93.5

Measurements of gauge lengths dataset: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, .396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Appendix C

R codes for PDF, CDF, quantile, mode, k-th order moment, skewness, excess kurtosis, pdf of order statistics, moments of order statistics and pseudo-random number generator are available at “github.com/PiotrSule/NEELS”.


THANK
YOU